

SYMMETRIC ABSTRACT HYPERGEOMETRIC POLYNOMIALS

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ABSTRACT. Consider an abstract operator L which acts on monomials x^n according to $Lx^n = \lambda_n x^n + \nu_n x^{n-2}$ for λ_n and ν_n some coefficients. Let $P_n(x)$ be eigenpolynomials of degree n of L : $LP_n(x) = \lambda_n P_n(x)$. A classification of all the cases for which the polynomials $P_n(x)$ are orthogonal is provided. A general derivation of the algebras explaining the bispectrality of the polynomials is given. The resulting algebras prove to be central extensions of the Askey-Wilson algebra and its degenerate cases.

1. INTRODUCTION

There is a much interest in obtaining abstractly a classification of orthogonal polynomials (OPs) that possess certain general properties. This indeed allows to characterize many families of OPs simultaneously and to shed light on the structural origin of features that they share. We here pursue in this respect a program undertaken recently [28], [27], [30] that aims to determine the orthogonal polynomials that are eigenfunctions of an operator L that has a prescribed action in monomial or Newtonian bases.

The simplest case, discussed in [28], is one where L acts according to $Lx^n = \lambda_n x^n + \rho_n x^{n-1}$ on monomials (μ_n and ρ_n are coefficients). We here examine the situation when the operator L rather acts as follows

$$(1.1) \quad Lx^n = \lambda_n x^n + \nu_n x^{n-2}, \quad n = 0, 1, 2, \dots$$

with λ_n and ν_n sets of real factors. All eigenpolynomials of L that form orthogonal ensembles will be determined and found to be the generalized Gegenbauer and Hermite polynomials and their q -analogs. Expectedly, these are symmetric polynomials.

The algebras encoding the bispectrality of these families shall also be obtained in an abstract fashion and seen to involve the reflection operator as a central element.

The outline of the paper is as follows. Some background is provided in Section 2 where the essential elements of the analysis presented in [28] will be reviewed. These results will prove useful to obtain in Section 3 the classification of the (necessarily) symmetric eigenpolynomials of the operator L defined by eq. (1.1). Section 4 is dedicated to the differential - difference realizations of the operators L that emerge in the classification and Section 5 focuses on the construction of the nonlinear algebras associated to these polynomials. The paper ends with concluding remarks.

2. BACKGROUND

Let L be an operator which transforms any polynomial of degree $n = 0, 1, 2, \dots$ to a polynomial of the same degree n . This operator preserves the $(n+1)$ -dimensional linear space of polynomials with degrees $\leq n$ and hence the eigenvalue problem for L on that space is well-posed.

To specify the operator L , we assume that its action on monomials is the following

$$(2.1) \quad Lx^n = \lambda_n x^n + \nu_n^{(1)} x^{n-1} + \nu_n^{(2)} x^{n-2} + \dots + \nu_n^{(J)} x^{n-J},$$

where $J = 1, 2, \dots$ is a fixed positive integer and where $\lambda_n, \nu_n^{(i)}$ are some coefficients. The eigenvalue problem will then read as follows in terms of monic polynomials $P_n(x) = x^n + O(x^{n-1})$:

$$(2.2) \quad LP_n(x) = \lambda_n P_n(x), \quad n = 0, 1, 2, \dots$$

It is assumed in what follows that the eigenvalues λ_n are nondegenerate, i.e.

$$(2.3) \quad \lambda_n \neq \lambda_m, \quad \text{if } n \neq m.$$

It can easily be checked that under condition (2.3), the eigenpolynomial $P_n(x)$ defined by (2.2) is unique. Indeed, if one assumes that there exists another monic polynomial $\tilde{P}_m(x)$, $m \leq n$ with the same eigenvalue λ_n it follows that $m = n$. Hence there exists two monic polynomials $P_n(x)$ and $\tilde{P}_n(x)$ with the same eigenvalue λ_n ; their difference $\Delta P_n(x) = P_n(x) - \tilde{P}_n(x)$ is a polynomial of a smaller degree, say $j < n$ which again satisfy the eigenvalue equation $L\Delta P_n(x) = \lambda_n \Delta P_n(x)$. This is however impossible as it would contradict the condition (2.3) thus showing that P_n is unique.

The main problem is to determine when the polynomials $P_n(x)$, $n = 0, 1, 2, \dots$ form an orthogonal set. This is so if there exists a nondegenerate linear functional σ acting on the space of polynomials such that

$$(2.4) \quad \langle \sigma, P_n(x)P_m(x) \rangle = 0, \quad n \neq m$$

where $\langle \sigma, \pi(x) \rangle$ stands for the evaluation of the functional σ on the polynomial $\pi(x)$. The functional σ is completely determined by the moments

$$(2.5) \quad \langle \sigma, x^n \rangle = c_n, \quad n = 0, 1, 2, \dots$$

and is said nondegenerate if $\Delta_n \neq 0$, $n = 0, 1, 2, \dots$, where $\Delta_n = |c_{i+k}|_{i,k=0}^n$ are the Hankel determinants constructed from the moments. The conditions (2.5) are equivalent to the conditions

$$(2.6) \quad \langle \sigma, P_n(x)\pi(x) \rangle = 0,$$

where $\pi(x)$ is any polynomial of degree lesser than n . Equivalently, the polynomials $P_n(x)$ are orthogonal iff they satisfy the three-term recurrence relation [6]

$$(2.7) \quad P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x) = x P_n(x), \quad n = 1, 2, \dots$$

with some coefficients b_n, u_n . The linear functional σ is nondegenerate iff $u_n \neq 0$, $n = 1, 2, \dots$. When the coefficients b_n are real and $u_n > 0$, $n = 1, 2, \dots$, the polynomials are orthogonal with respect to a positive measure $d\mu(x)$ on the real axis [12]:

$$(2.8) \quad \int_a^b P_n(x)P_m(x)d\mu(x) = h_n \delta_{nm},$$

where $h_0 = 1$ and $h_n = u_1 u_2 \dots u_n$, $n = 1, 2, \dots$. The integration limits a, b in (2.8) may be either finite or infinite.

It is shown in [28] that the necessary and sufficient conditions for the polynomials $P_n(x)$ to be orthogonal are that

$$(2.9) \quad \langle \sigma, g(x)Lf(x) \rangle = \langle \sigma, f(x)Lg(x) \rangle$$

for any pair of polynomials $f(x), g(x)$. This means that L must be symmetric on the space of polynomials with respect to the functional σ . See also [7] where the same condition is derived when L is a higher-order difference operator. In fact, (2.9) implies the orthogonality of the eigenpolynomials for *any* linear operator L preserving polynomiality and such that $\deg(Lp(x)) \leq \deg(p(x))$ for any

polynomial $p(x)$. Choosing $f(x) = x^n$ and $g(x) = x^m$ and assuming that the operator L acts as in (2.1), condition (2.9) is seen to amount to

$$(2.10) \quad (\lambda_n - \lambda_m)c_{n+m} + (\nu_n^{(1)} - \nu_m^{(1)})c_{n+m-1} + \cdots + (\nu_n^{(J)} - \nu_m^{(J)})c_{n+m-J} = 0,$$

where n, m can take all admissible values (i.e. $n + m \geq J$).

When $J = 1$, (2.10) becomes

$$(2.11) \quad (\lambda_n - \lambda_m)c_{n+m} + (\nu_n^{(1)} - \nu_m^{(1)})c_{n+m-1} = 0.$$

From the no-degeneracy condition $\lambda_n \neq \lambda_m$ for all $n \neq m$, it follows [28] that

$$(2.12) \quad \frac{\lambda_{n+1} - \lambda_k}{\nu_{n+1} - \mu_k} = \frac{\lambda_n - \lambda_{k+1}}{\nu_n - \mu_{k+1}}$$

should be valid for all pairs of integers $k, n = 0, 1, 2, \dots$.

As shown in [28], all nondegenerate solutions of (2.12) are obtained as solutions of the linear difference equations:

$$(2.13) \quad \lambda_{n+1} + \lambda_{n-1} - \Omega\lambda_n + B_1 = 0, \quad \nu_{n+1}^{(1)} + \nu_{n-1}^{(1)} - \Omega\nu_n^{(1)} + B_2 = 0,$$

where Ω, B_1, B_2 are arbitrary parameters. The type of solution depends on the choice of the parameter Ω .

When $\Omega = q + q^{-1} > 2$ with q a real parameter, the general solution is

$$(2.14) \quad \lambda_n = \alpha_1 q^n + \alpha_2 q^{-n} + \alpha_0, \quad \nu_n^{(1)} = \beta_1 q^n + \beta_2 q^{-n} + \beta_0$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are arbitrary parameters and α_0, β_0 are related to B_1, B_2 as follows

$$(2.15) \quad B_1 = \alpha_0 q^{-1}(1 - q)^2, \quad B_2 = \beta_0 q^{-1}(1 - q)^2.$$

This case corresponds to the little q -Jacobi polynomials and their “specializations”: q -Laguerre, little q -Laguerre, alternative q -Charlier and Stieltjes-Wigert polynomials.

When $\Omega = 2$ the general solution is

$$(2.16) \quad \lambda_n = \alpha_2 n^2 + \alpha_1 n + \alpha_0, \quad \nu_n^{(1)} = \beta_2 n^2 + \beta_1 n + \beta_0$$

where $\alpha_0, \alpha_1, \beta_1, \beta_0$ are arbitrary constants and

$$(2.17) \quad B_1 = -2\alpha_2, \quad B_2 = -2\beta_2.$$

This case corresponds to the Jacobi and Laguerre polynomials.

Finally, when $\Omega = -2$,

$$(2.18) \quad \lambda_n = \alpha_1(-1)^n + \alpha_2(-1)^n n + \alpha_0, \quad \nu_n^{(1)} = \beta_1(-1)^n + \beta_2(-1)^n n + \beta_0$$

with α_1, α_2 arbitrary and

$$(2.19) \quad B_1 = -4\alpha_0, \quad B_2 = -4\beta_0.$$

This case corresponds to the little -1 Jacobi polynomials.

The case $\Omega < -2$ is basically obtained from the one when $\Omega > 2$ under the substitution $q \rightarrow -q$. The remaining case $|\Omega| < 2$ requires that q belongs to the unit circle $|q| = 1$; the corresponding orthogonal polynomials are not positive definite [28] and will not be considered here. It should be stressed that the parameters α_i and β_i in the above formulas can be arbitrary. However, from the condition that $L(1) = \lambda_0$ is a constant it follows that necessarily $\nu_0^{(1)} = 0$. This leads to obvious restrictions on the parameters α_i, β_i .

3. THE CLASSIFICATION PROBLEM AND ADMISSIBILITY CONDITIONS

We have reviewed above the results of the classification of all the orthogonal eigenpolynomials of L as given in (2.1) when $J = 1$. In order of complexity, the next classification problem corresponds to $J = 2$. In this case, the necessary and sufficient conditions (2.10) for the polynomials $P_n(x)$ to be orthogonal specialize to

$$(3.1) \quad (\lambda_n - \lambda_m)c_{n+m} + (\nu_n^{(1)} - \nu_m^{(1)})c_{n+m-1} + (\nu_n^{(2)} - \nu_m^{(2)})c_{n+m-2} = 0.$$

This system of equations proves rather complicated and we shall restrict ourselves to the simpler case that result when $\nu_n^{(1)} = 0$. This means that we will study the operator L which acts as follows

$$(3.2) \quad Lx^n = \lambda_n x^n + \nu_n x^{n-2}$$

with two unknown coefficients λ_n, ν_n (we omit the superscript in ν_n). We assume that λ_n satisfy the nondegenerate condition (2.3) and moreover that

$$(3.3) \quad \nu_0 = \nu_1 = 0, \quad \nu_n \neq 0, \quad n = 2, 3, 4, \dots$$

The necessary and sufficient conditions for orthogonality will now read:

$$(3.4) \quad (\lambda_n - \lambda_m)c_{n+m} + (\nu_n - \nu_m)c_{n+m-2} = 0.$$

The classification problem that we will tackle now amounts to finding all the orthogonal polynomials that are eigenfunctions of an operator of the form specified by (3.2).

We first prove the following simple lemma.

Lemma 1. *The monic eigenpolynomials $P_n(x) = x^n + O(x^{n-1})$ are uniquely determined by the equation $LP_n(x) = \lambda_n P_n(x)$ if the eigenvalues λ_n are nondegenerate. Moreover, these eigenpolynomials are symmetric, i.e. they satisfy the parity condition*

$$(3.5) \quad P_n(-x) = (-1)^n P_n(x).$$

We already proved that the polynomial $P_n(x)$ is unique if the eigenvalues λ_n are nondegenerate. Let us introduce the parity operator R that acts on any function $f(x)$ as

$$(3.6) \quad Rf(x) = f(-x).$$

For all monomials it is clear that

$$(3.7) \quad LRx^n = RLx^n = (-1)^n (\lambda_n x^n + \nu_n x^{n-2}).$$

Hence the operator R commutes with L

$$(3.8) \quad LR = RL$$

on the space of all polynomials.

This means that the operator R leaves any eigenpolynomial $P_n(x)$ in the same eigenspace. Because of the uniqueness of the polynomial $P_n(x)$, it follows that the various eigensubspaces are one-dimensional and hence

$$(3.9) \quad RP_n(x) = \pm P_n(x).$$

In turn, condition (3.9) means that the polynomial $P_n(x)$ should be an even or an odd function. We hence arrive at (3.5).

This condition implies that every even polynomial $P_{2n}(x)$ contains only the even monomials x^{2k} while every odd polynomial $P_{2n+1}(x)$ contains only the odd monomials x^{2k+1} :

$$(3.10) \quad P_{2n}(x) = U_n(x^2), \quad P_{2n+1}(x) = xV_n(x^2),$$

where $U_n(x), V_n(x)$ are monic polynomials of degree n . Such orthogonal polynomials are called symmetric [6].

Assume now that the polynomials $P_n(x)$ are orthogonal with corresponding moments c_n . Condition (3.5) means that all odd moments are zero $c_{2n+1} = 0$, $n = 0, 1, \dots$. This is equivalent to the statement that in the recurrence relation (2.7), the diagonal recurrence coefficients b_n vanish [6] and thus the symmetric orthogonal polynomials satisfy the recurrence relation

$$(3.11) \quad P_{n+1}(x) + u_n P_{n-1}(x) = x P_n(x).$$

Assume that the polynomials $P_n(x)$ have the following expansion

$$(3.12) \quad P_n(x) = \sum_{k=0}^n A_{nk} x^k.$$

The coefficients A_{nk} have the property that $A_{nk} = 0$ when the parities of n and k are different. This follows from (3.10). Moreover, $A_{nn} = 1$ because the polynomials $P_n(x)$ are monic.

Substituting (3.12) into the recurrence relation (3.11), we find on the one hand, the system of conditions

$$(3.13) \quad A_{n+1,k} + u_n A_{n-1,k} = A_{n,k-1}, \quad k = 0, 1, \dots, n-1.$$

For $k = n-1$, we obtain from (3.13) the expression for u_n :

$$(3.14) \quad u_n = A_{n+1,n-1} - A_{n,n-2}.$$

On the other hand, we have from the eigenvalue equation $LP_n(x) = \lambda_n P_n(x)$ that

$$(3.15) \quad A_{nk}(\lambda_n - \lambda_k) + \nu_k A_{n,k+2} = 0, \quad k = n-2, n-4, n-6, \dots$$

This relation allows to write explicitly the coefficients A_{nk} in terms of λ_n, ν_n :

$$(3.16) \quad A_{n,n-2} = -\frac{\nu_n}{\lambda_n - \lambda_{n-2}}, \quad A_{n,n-4} = \frac{\nu_n \nu_{n-2}}{(\lambda_n - \lambda_{n-2})(\lambda_n - \lambda_{n-4})}, \dots$$

In particular, comparing (3.16) with (3.14), we arrive at the following expression for the recurrence coefficient u_n in terms of λ_n, ν_n :

$$(3.17) \quad u_n = \frac{\nu_n}{\lambda_n - \lambda_{n-2}} - \frac{\nu_{n+1}}{\lambda_{n+1} - \lambda_{n-1}}.$$

Moreover, an elementary analysis shows (see, e.g. [6]) that the polynomials $U_n(x)$ and $V_n(x)$ are again orthogonal having the moments

$$(3.18) \quad c_n^{(U)} = c_{2n}, \quad c_n^{(V)} = c_{n+1}^{(U)} = c_{2n+2}, \quad n = 0, 1, 2, \dots$$

It also follows that the polynomials $V_n(x)$ are the kernel polynomials of the polynomials $U_n(x)$ [6]:

$$(3.19) \quad V_n(x) = \frac{U_{n+1}(x) - A_n U_n(x)}{x}, \quad A_n = \frac{U_{n+1}(0)}{U_n(0)}.$$

Let us introduce the coefficients

$$(3.20) \quad \lambda_n^{(U)} = \lambda_{2n}, \quad \lambda_n^{(V)} = \lambda_{2n+1}, \quad \nu_n^{(U)} = \nu_{2n}, \quad \nu_n^{(V)} = \nu_{2n+1}, \quad n = 0, 1, 2, \dots$$

It follows from (3.4) that

$$(3.21) \quad (\lambda_n^{(U)} - \lambda_m^{(U)})c_{n+m}^{(U)} + (\nu_n^{(U)} - \nu_m^{(U)})c_{n+m-1}^{(U)} = 0$$

and

$$(3.22) \quad (\lambda_n^{(V)} - \lambda_m^{(V)})c_{n+m}^{(V)} + (\nu_n^{(V)} - \nu_m^{(V)})c_{n+m-1}^{(V)} = 0.$$

Comparing conditions (3.21), (3.22) with (2.11), we see that both polynomials $U_n(x)$ and $V_n(x)$ are abstract hypergeometric orthogonal polynomials of the $J = 1$ type and must hence belong to one of the three classes described in Section 2. This yields explicit expressions for the moments $c_n^{(U)}, c_n^{(V)}$ and hence for the moments c_n . One should not lose sight however that according to (3.18), c_n^U and c_n^V collectively provide the moments of the polynomials $P_n(x)$ and that in view of (3.4), we have the following additional restrictions:

$$(3.23) \quad \frac{\lambda_n^{(V)} - \lambda_m^{(V)}}{\nu_n^{(V)} - \nu_m^{(V)}} = \frac{\lambda_{n+1}^{(U)} - \lambda_m^{(U)}}{\nu_{n+1}^{(U)} - \nu_m^{(U)}}.$$

It is not difficult to determine the solutions to these conditions. The results can be summarized as follows:

(i) Both polynomials $U_n(x)$ and $V_n(x)$ belong to the little q -Jacobi class or to its special and degenerate cases (say, q -Laguerre polynomials); generically, for little q -Jacobi polynomials we choose:

$$(3.24) \quad \begin{aligned} \lambda_n^{(U)} &= (1 - q^{2n})(q^{-2n} + a) + \varepsilon_0, \quad \lambda_n^{(V)} = \rho(q^{2n} - 1)(q^{-2n} + aq^2) + \varepsilon_1, \\ \nu_n^{(U)} &= (q^{2n} - 1)(q^{-2n} + b), \quad \nu_n^{(V)} = \rho(1 - q^{2n})(q^{-2n} + bq^2) \end{aligned}$$

where $\rho, \varepsilon_0, \varepsilon_1$ are arbitrary real parameters (ρ is assumed to be nonzero). Note that the expressions for $\lambda_n^{(V)}, \nu_n^{(V)}$ are slightly different from those for $\lambda_n^{(U)}, \nu_n^{(U)}$; this is due to condition (3.23).

The meaning of the free parameters $\rho, \varepsilon_0, \varepsilon_1$ is the following. Assume that L is an admissible operator, i.e. that the coefficients λ_n and ν_n correspond to symmetric orthogonal polynomials $P_n(x)$ which are eigenfunctions of the operator L . We can introduce a more general operator

$$(3.25) \quad \tilde{L} = \xi_1 L + \xi_2 LR + \eta_1 \mathcal{I} + \eta_2 R,$$

where \mathcal{I} is the identity operator. The reflection operator R commutes with L . Hence the new operator \tilde{L} will again be an admissible operator for the same symmetric orthogonal polynomials $P_n(x)$ but with different eigenvalue:

$$(3.26) \quad \tilde{L}P_n(x) = \tilde{\lambda}_n P_n(x)$$

where

$$(3.27) \quad \tilde{\lambda}_n = (\xi_1 + (-1)^n \xi_2) \lambda_n + \eta_1 + \eta_2 (-1)^n.$$

Among these 4 parameters, ξ_1 can be considered as a common factor and we can always put $\xi_1 = 1$. The other three constants ξ_2, η_1, η_2 , correspond to the three parameters $\rho, \varepsilon_0, \varepsilon_1$ in (3.24). In what follows we will use the freedom in the choice of these parameters in order to obtain the most simple presentations of the algebra associated to the various families of symmetric polynomials (see Section 5).

From (3.24) and (3.17), the explicit expression for the recurrence coefficient u_n are found to be:

$$(3.28) \quad u_{2n} = \frac{q^{2n}(1 - q^{2n})(aq^{2n-2} - b)}{(1 + aq^{4n})(1 + aq^{4n-2})}, \quad u_{2n+1} = \frac{q^{2n}(bq^{2n+2} + 1)(aq^{2n} + 1)}{(1 + aq^{4n})(1 + aq^{4n+2})}.$$

As expected, these recurrence coefficients correspond to those of the generalized little q -Jacobi polynomials [9].

Instead of classifying all degenerate cases of the generalized little q -Jacobi polynomials we consider only the one with

$$(3.29) \quad \begin{aligned} \lambda_n^{(U)} &= (1 - q^n) + \varepsilon_0, \quad \lambda_n^{(V)} = \rho(1 - q^n) + \varepsilon_1, \\ \nu_n^{(U)} &= (1 - q^n)(q^{-n} - b), \quad \nu_n^{(V)} = \rho(1 - q^n)(q^{-n} - bq) \end{aligned}$$

As in the previous situation, we have for the recurrence coefficients the expression

$$(3.30) \quad u_{2n} = bq^{2-2n}(q^{2n} - 1), \quad u_{2n+1} = q^{-4n-2}(q^2 + 1 - q^{2n+2})(bq^{2n+2} + 1).$$

These recurrence coefficients correspond to those of the generalized q -Laguerre polynomials [9].

(ii) Both polynomials $U_n(x)$ and $V_n(x)$ are either Jacobi or Laguerre polynomials. Namely, for the Jacobi polynomials

$$(3.31) \quad \begin{aligned} \lambda_n^{(U)} &= -2n(2n+a) + \varepsilon_0, & \lambda_n^{(V)} &= 2\rho n(2n+a+2) + \varepsilon_1, \\ \nu_n^{(U)} &= 2n(2n+b), & \nu_n^{(V)} &= -2\rho n(2n+b+2). \end{aligned}$$

According to (3.17) this leads to the recurrence coefficients

$$(3.32) \quad u_{2n} = 2 \frac{n(2n-2+a-b)}{(a+4n)(a+4n-2)}, \quad u_{2n+1} = \frac{(2n+a)(2n+2+b)}{(a+4n+2)(a+4n)}$$

which correspond to those of the generalized Gegenbauer polynomials [6].

For the Laguerre class we have similarly

$$(3.33) \quad \begin{aligned} \lambda_n^{(U)} &= -2n + \varepsilon_0, & \lambda_n^{(V)} &= 2\rho n + \varepsilon_1, \\ \nu_n^{(U)} &= 2n(2n+b), & \nu_n^{(V)} &= -2\rho n(2n+b+2) \end{aligned}$$

which leads to the recurrence coefficients

$$(3.34) \quad u_{2n} = 2n, \quad u_{2n+1} = 2n+2+b.$$

They correspond to the generalized Laguerre polynomials. For the special case $b = -1$, we obtain the Hermite polynomials.

We have thus described all the admissible classes of symmetric hypergeometric orthogonal polynomials. Note that in contrast to the classification scheme presented in [28], only two classes exist for the operator L of the form (3.2), namely the q -Jacobi and the Jacobi ones. The third class, that of the little -1 Jacobi polynomials, is absent in our classification scheme. The reason for this is the following. The polynomials $U_n(x)$ and $V_n(x)$ introduced in (3.10) are both classical orthogonal polynomials related by the Christoffel transform (3.19). It is well known that for the ordinary Jacobi and for the little q -Jacobi polynomials, the Christoffel transform (3.19) leads to polynomials in the same class but with slightly modified parameters. However, for the little -1 Jacobi polynomials, the Christoffel transform of the polynomials $U_n(x)$ gives non-classical polynomials that do not belong to the little -1 Jacobi family [26]. This means, in particular, that the compatibility condition (3.23) does not hold in this case.

4. REALIZATIONS OF THE OPERATOR L

The admissible operators L lead to symmetric orthogonal polynomials $P_n(x)$ which are formally bispectral, i.e. they satisfy the “dual” eigenvalue equations (2.2) and (3.11). For the classical orthogonal polynomials of the Askey tableau [15], the operator L can be presented in terms of either differential or q -difference operators. We now examine this question for the operators L that have arisen in the present study. In contrast to the classical case, here the operator L involves the reflection operator R which means that we deal with the operators of Dunkl type [26].

Consider first the case of the generalized Jacobi polynomials. We have the explicit expressions (3.24) for the coefficients λ_n, ν_n of the operator L . Introduce the projection operators

$$(4.1) \quad \pi_0 = \frac{1}{2}(1+R), \quad \pi_1 = \frac{1}{2}(1-R).$$

The operator π_0 projects any function onto the subspace of even functions while the operator π_1 projects onto the subspace of odd functions.

It is easy to show that the operator L can be presented in the form

$$(4.2) \quad L = L_0\pi_0 + L_1\pi_1,$$

where the operators L_0 and L_1 are both 4-difference operators corresponding to the little q -Jacobi polynomials. In more details, let us introduce the operators T_q^+ and T_q^- defined by the following actions:

$$(4.3) \quad T_q^+ f(x) = f(xq), \quad T_q^- f(x) = f(x/q)$$

Then

$$(4.4) \quad L_0 = (-a + bx^{-2})(T_q^+ - \mathcal{I}) + (1 - x^{-2})(T_q^- - \mathcal{I}) + \varepsilon_0$$

and

$$(4.5) \quad L_1 = \rho q(a - bx^{-2})T_q^+ + \rho q(-1 + x^{-2})T_q^- + \varepsilon_1 + \rho - a\rho q^2 + \rho(bq^2 - 1)x^{-2}.$$

These expressions for the operators L_0 and L_1 can easily be verified through their actions on the monomials x^n . Indeed, we have

$$(4.6) \quad L_0 x^{2n} = \lambda_{2n} x^{2n} + \nu_{2n} x^{2n-2}$$

and

$$(4.7) \quad L_1 x^{2n+1} = \lambda_{2n+1} x^{2n+1} + \nu_{2n+1} x^{2n-1}.$$

Combining, we can present these formulas as

$$(4.8) \quad Lx^n = (L_0\pi_0 + L_1\pi_1)x^n = \lambda_n x^n + \nu_n x^{n-2}$$

which means that our operator L satisfies the required conditions.

The pure “classical” situation arises when $L_0 = L_1$. In this case (and only in this case) the reflection operator R vanishes in the expression of the operator L as seen from the definition of the projection operators (4.1). It is observed from (4.4)-(4.5) that the condition $L_0 = L_1$ can be achieved only if

$$(4.9) \quad b = \rho = -q^{-1}, \quad \varepsilon_1 = \varepsilon_0 + (q^{-1} - 1)(aq + 1).$$

Given (4.9), the recurrence coefficients (3.28) become

$$(4.10) \quad u_n = \frac{q^{n-1}(1 - q^n)(1 + aq^{n-1})}{(1 + aq^{2n})(1 + aq^{2n-2})}.$$

It is possible to identify the orthogonal polynomials with recurrence coefficients given by (4.10) as a special case of the symmetric big q -Jacobi polynomials. Indeed, the big q -Jacobi polynomials $P_n(x; \alpha, \beta, \gamma; q)$ depend on 3 parameters α, β, γ (using the notation of [15]) and the recurrence coefficients (4.10) correspond to the choice $a = q\alpha^2, \beta = -\gamma = \alpha$.

Consider now the generalized Gegenbauer polynomials. Applying the same approach, we arrive at the operator (4.2) with

$$(4.11) \quad L_0 = (1 - x^2)\partial_x^2 + ((b+1)x^{-1} - (a+1)x)\partial_x + \varepsilon_0$$

and

$$(4.12) \quad L_1 = \rho(x^2 - 1)\partial_x^2 + \rho((a+1)x - (b+1)x^{-1})\partial_x + \varepsilon_1 - \rho(a+1) + \rho(b+1)x^{-2}.$$

In the special case when $b = -1$ and $\rho = -1, \varepsilon_0 = \varepsilon_1 = a + 1$, we see that $L_1 = L_0$ and hence the operator L becomes classical as it does not contain the reflection operator R . This corresponds to the ordinary Gegenbauer polynomials. Indeed, when $b = -1$ one sees from (3.32) that

$$(4.13) \quad u_n = \frac{n(n+a-1)}{(2n+a)(2n+a-2)}$$

thus recovering the recurrence coefficients of the Gegenbauer polynomials $C_n^{(a/2)}(x)$ (in the notation of [15]).

For general values of the parameters a and b the operator L is not classical. In [4] the operator L for the generalized Gegenbauer polynomials was presented as a second order Dunkl differential operator.

Consider finally the generalized Laguerre polynomials corresponding to the choice (4.1). Again it is easily seen that

$$(4.14) \quad L_0 = \partial_x^2 + (-x + (b+1)x^{-1})\partial_x + \varepsilon_0$$

and that

$$(4.15) \quad L_1 = -\rho\partial_x^2 + \rho(x - (b+1)x^{-1})\partial_x + \varepsilon_1 - \rho + \rho(b+1)x^{-2}.$$

When $b = -1$ and $\rho = -1, \varepsilon_0 = \varepsilon_1 + 1$, the operators L_1 and L_0 coincide. This corresponds to the ordinary Hermite polynomials. For generic values of b the operator L can be reduced to a second order differential operator of Dunkl type [4].

We thus have constructed explicit presentations of the operator L in terms of differential or q -difference operators. In the general case, the operator L is non-classical as it contains the reflection operator R ; only in some special instances (those of the Gegenbauer and Hermite polynomials) can the operator L be reduced to a classical one.

5. ASSOCIATED ALGEBRAS

We shall now obtain in our abstract formalism the algebras that underscore the bispectrality of the polynomials that have been identified. In view of the definite parity properties of these polynomials, it will prove pertinent to use the operator of multiplication by the square of the argument x :

$$(5.1) \quad Y = x^2$$

in conjunction with L as generators. Define

$$(5.2) \quad W^{(1)} = L^2Y + YL^2 - \Omega LY L, \quad W^{(2)} = LY^2 + Y^2L - \Omega YLY,$$

where $\Omega = q^2 + q^{-2}$ for the little q -Jacobi class and $\Omega = 2$ for the Jacobi family (including degenerate cases). We shall take $W^{(1)}$ and $W^{(2)}$ to provide the left hand sides of the defining relations of the algebras. Consider the following q -commutator

$$(5.3) \quad [A, B]_q = qAB - q^{-1}BA.$$

Then $W^{(1)}$ and $W^{(2)}$ can be presented in the form

$$(5.4) \quad W^{(1)} = [L, [L, Y]_q]_{q^{-1}}, \quad W^{(2)} = [Y, [Y, L]_q]_{q^{-1}}$$

and they obviously become double commutators when $q = 1$ and $\Omega = 2$. In order to obtain the defining relations in the abstract context, we could determine the action of $W^{(1)}$ and $W^{(2)}$ on monomials and use the relation (3.2) to find the identities we are looking for. Alternatively, we can

use the explicit representations of the operators L and Y in terms of the differential or q -difference operators obtained in the previous section. We shall opt for the latter course in the following.

We start with the case of the generalized little q -Jacobi polynomials. There are three independent parameters, $\varepsilon_0, \varepsilon_1, \rho$, in the expression of the coefficients λ_n, ν_n in (3.24). The orthogonal polynomials $P_n(x)$ do not depend on these constants; only the concrete expression of the operator L will involve these parameters. Their origin can be traced back to formula (3.25) that gives the operator L as a linear combination (with arbitrary factors) of basic constituents. We can always use the latitude offered by these free constants to simplify the presentations. In the case of the generalized little q -Jacobi polynomials it is convenient to put

$$(5.5) \quad \rho = -q^{-1}, \quad \varepsilon_0 = 1 - a, \quad \varepsilon_1 = q^{-1} - aq$$

We then have the relations

$$(5.6) \quad [Y, [Y, L]_q]_{q^{-1}} = Y(\xi_0 + \xi_1 R)$$

and

$$(5.7) \quad [L, [L, Y]_q]_{q^{-1}} = L(\xi_0 + \xi_1 R) + \eta Y + \zeta \mathcal{I},$$

where R is the reflection operator (3.6) and where

$$(5.8) \quad \begin{aligned} \xi_0 &= \frac{1}{2}(q-1)(b-q^{-1})(q-q^{-1})^2, & \xi_1 &= \frac{1}{2}(1-q)(b+q^{-1})(q-q^{-1})^2, \\ \eta &= a(q^2 - q^{-2})^2, & \zeta &= -(a+bq^2)(q^2 - q^{-2})^2. \end{aligned}$$

It is seen that the operators L and Y constitute a special case of a special central extension of the Askey-Wilson algebra $AW(3)$. The central element is the reflection operator R which commutes with both L and Y . Koornwinder in [16] has introduced and studied the central extension $AW(3, Q_0)$ of the generic Askey-Wilson algebra. Terwilliger has also considered a similar algebraic construction and has called it the “universal Askey-Wilson algebra” [22]. In this scheme, the central element is an operator T satisfying the algebraic relation

$$(5.9) \quad (T + \alpha\beta)(T + 1) = 0,$$

where α, β are arbitrary parameters. If one assumes that $\alpha\beta \neq 1$ (the generic situation), by an affine transformation the operator T can be reduced to the reflection operator R which satisfy the algebraic relation $R^2 = \mathcal{I}$. We thus see that the algebra (5.6) and (5.7) can be considered as a special case of $AW(3, Q_0)$ [16], [22]. In the conventional approach, the algebra $AW(3, Q_0)$ is related to the non-symmetric Askey-Wilson polynomials which are Laurent polynomials with nontrivial orthogonality properties and only in the symmetric case do these polynomials reduce to the ordinary Askey-Wilson polynomials (see [16] for details). Here, however, the algebra $AW(3, Q_0)$ describes *ordinary* orthogonal polynomials which exhibit a nontrivial bispectral property with respect to the operator L .

There is also a relation with a degenerate q -Onsager algebra [13]. Indeed, taking an extra commutators will bring (5.6) and (5.7) to the relations

$$(5.10) \quad [Y, [Y, [Y, L]_q]_{q^{-1}}] = 0, \quad [L, [L, [L, Y]_q]_{q^{-1}}] = \eta[L, Y]$$

which correspond to those of a degenerate q -Onsager algebra [13]. Note that the q -Onsager algebra at roots of unity were seen to provide an underpinning for the complementary Bannai-Ito polynomials. [3].

Observe that when $b = -q^{-1}$, the terms with the operator R vanish and our algebra becomes a special case of the ordinary Askey-Wilson algebra. In the previous section we already mentioned

that this condition corresponds to a special case of the big q -Jacobi polynomials that leads to classical polynomials. In this case, the occurrence of the standard $AW(3)$ algebra is rather natural. In general, it is a central extension of $AW(3)$ that appears.

For the generalized Gegenbauer polynomials we put

$$(5.11) \quad \rho = -1, \quad \varepsilon_0 = 1 - a^2/4, \quad \varepsilon_1 = -a - a^2/4$$

in (4.14)-(4.15). This brings the following commutation relations

$$(5.12) \quad [Y, [Y, L]] = -8Y^2 + 8Y, \quad [L, [L, Y]] = -8\{L, Y\} + 8L + 4(b+1)R + \mu\mathcal{I}$$

where $\{L, Y\} = LY + YL$ stands for the anticommutator and where

$$\mu = 2a^2 - 4ab - 8a + 4b + 4.$$

In this case the algebra (5.12) is a central extension of the quadratic Jacobi algebra introduced in [10].

As we already demonstrated, when $b = -1$, the polynomials $P_n(x)$ become the classical Gegenbauer polynomials. With this choice for the parameter b , the term containing the reflection operator R disappears in (5.12) and we return to the ordinary Jacobi algebra.

Finally, for the generalized Hermite polynomials, it is convenient to take the parameters of the operator L as follows:

$$(5.13) \quad \varepsilon_0 = -1 - b/2, \quad \varepsilon_1 = -2 - b/2, \quad \rho = -1.$$

We then obtain the relations

$$(5.14) \quad [Y, [Y, L]] = 8Y, \quad [L, [L, Y]] = 8L + 4Y$$

which define an algebra isomorphic to sl_2 . It is interesting to note that even in the case of nonclassical generalized Laguerre polynomials (i.e. when $b \neq -1$), the algebra generated by the operators Y and L , is a Lie algebra. This can be achieved by an appropriate choice of the parameters $\varepsilon_0, \varepsilon_1, \rho$. For all other cases of nonclassical polynomials, the appearance of a central extension of the “classical” bispectral algebras (like $AW(3)$ and the Jacobi algebra) is inevitable.

6. CONCLUDING REMARKS

In summary this paper has offered a classification of all positive-definite orthogonal polynomials that are eigenfunctions of an operator L acting as $Lx^n = \lambda_n x^n + \nu_n x^{n-2}$ on monomials. This abstract approach has provided a synthetic description of the many symmetric semiclassical orthogonal q -polynomials that have been identified recently [9] (their $q = 1$ versions have also been included). For concision, we have taken $0 < q \leq 1$.

The general form of the non-linear algebras that are associated to the polynomials arising in the classification have also been obtained. Interestingly, these algebras extend those of the Askey-Wilson type in that they involve an involution ($R^2 = 1$). Algebras with a similar feature have already been encountered in the theory of complementary Bannai-Ito polynomials [8] and of the dual -1 Hahn polynomials [23]. It has recently been found that in these cases, the corresponding algebras can be embedded in q -Onsager algebras for q a root of unity [3]. It would be interesting to provide a similar connection between tridiagonal pairs and the algebras identified in this paper. We plan on returning to this question in the future.

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